

A Four Function Variational Principle for Barotropic Magnetohydrodynamics

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Abstract

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. In a previous work Yahalom & Lynden-Bell introduced a simpler Eulerian variational principles from which all the relevant equations of magnetohydrodynamics can be derived. The variational principle was given in terms of six independent functions for non-stationary flows and three independent functions for stationary flows. This is less than the seven variables which appear in the standard equations of magnetohydrodynamics which are the magnetic field \vec{B} the velocity field \vec{v} and the density ρ . In this work I will improve on the previous results showing that non-stationary magnetohydrodynamics should be described by four functions .

1 Introduction

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [1] has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov and Moffatt [2] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of magnetohydrodynamics which are the magnetic field \vec{B} the velocity field \vec{v} and the density ρ . Kats [3] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. Moreover, Kats have shown that the variables he

suggested can be utilized to describe the motion of arbitrary discontinuity surfaces [4, 5]. Sakurai [6] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [1]. A method of solving the equations for those two variables was introduced by Yang, Sturrock & Antiochos [7]. In a recent work Yahalom & Lynden-Bell [8, 9] have combined the Lagrangian of Sturrock [1] with the Lagrangian of Sakurai [6] to obtain an **Eulerian** Lagrangian principle depending on only six functions. The vanishing of the variational derivatives of this Lagrangian entail all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resemble the equations of Frenkel, Levich & Stilman [10] (see also [11]). Furthermore, it was shown that for stationary flows three functions will suffice in order to describe a Lagrangian principle for barotropic magnetohydrodynamics. The non-singlevaluedness of the functions appearing in the reduced representation of barotropic magnetohydrodynamics was discussed in particular with connection to the topological invariants of magnetic and cross helicities. It was shown how the conservation of cross helicity can be easily generated using the Noether theorem and the variables introduced in that paper. In the current paper I improve on the previous results and show that four functions are enough to describe a general non stationary barotropic magnetohydrodynamics, the idea is borrowed from [12] see also [13, 14, 15].

The plan of this paper is as follows: First I introduce the standard notations and equations of barotropic magnetohydrodynamics. Next I introduce the potential representation of the magnetic field \vec{B} and the velocity field \vec{v} . This is followed by a review of the Eulerian variational principle developed by Yahalom & Lynden-Bell [8, 9]. After those introductory sections I will present the four function Eulerian variational principles for non-stationary magnetohydrodynamics.

2 The standard formulation of barotropic magnetohydrodynamics

The standard set of equations solved for barotropic magnetohydrodynamics are given below:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}), \quad (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (2)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (3)$$

$$\rho \frac{d\vec{v}}{dt} = \rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right) = -\vec{\nabla} p(\rho) + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{4\pi}. \quad (4)$$

The following notations are utilized: $\frac{\partial}{\partial t}$ is the temporal derivative, $\frac{d}{dt}$ is the temporal material derivative and $\vec{\nabla}$ has its standard meaning in vector calculus. \vec{B} is the magnetic field vector, \vec{v} is the velocity field vector and ρ is the fluid density. Finally $p(\rho)$ is the pressure which we assume depends on the density alone (barotropic case). The justification for those equations and the conditions under which they apply can be found in standard books on magnetohydrodynamics (see for example [1]). Equation (1) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), equation (2) describes the fact that the magnetic field is solenoidal, equation (3) describes the conservation of mass and equation (4) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. The term:

$$\vec{J} = \frac{\vec{\nabla} \times \vec{B}}{4\pi}, \quad (5)$$

is the electric current density which is not connected to any mass flow. The number of independent variables for which one needs to solve is seven (\vec{v}, \vec{B}, ρ) and the number of equations (1,3,4) is also seven. Notice that equation (2) is a condition on the initial \vec{B} field and is satisfied automatically for any other time due to equation (1). Also notice that $p(\rho)$ is not a variable rather it is a given function of ρ .

3 Potential representation of vector quantities of magnetohydrodynamics

It was shown in [8] that \vec{B} and \vec{v} can be represented in terms of five scalar functions $\alpha, \beta, \chi, \eta, \nu$. Following Sakurai [6] the magnetic field takes the form:

$$\vec{B} = \vec{\nabla} \chi \times \vec{\nabla} \eta. \quad (6)$$

Hence \vec{B} satisfies automatically equation (2) and is orthogonal to both $\vec{\nabla} \chi$ and $\vec{\nabla} \eta$. A similar representation was suggested by Dungey [16] but not in the context of variational analysis. The above expression can also describe a magnetic field with non-zero magnetic helicity as was demonstrated in [8].

Moreover, the velocity \vec{v} can be represented in the following form:

$$\vec{v} = \vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta. \quad (7)$$

this representation is a generalization of the Clebsch representation [17] for magnetohydrodynamics.

4 The Action of Barotropic Magnetohydrodynamics

It was shown in [8] that the action of barotropic magnetohydrodynamics takes the form:

$$\begin{aligned} A &\equiv \int \mathcal{L} d^3x dt, \\ \mathcal{L} &\equiv -\rho \left[\frac{\partial \nu}{\partial t} + \alpha \frac{\partial \chi}{\partial t} + \beta \frac{\partial \eta}{\partial t} + \varepsilon(\rho) + \frac{1}{2}(\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta)^2 \right] \\ &\quad - \frac{1}{8\pi}(\vec{\nabla}\chi \times \vec{\nabla}\eta)^2, \end{aligned} \quad (8)$$

in which $\varepsilon(\rho)$ is the specific internal energy. Taking the variational derivatives to zero for arbitrary variations leads to the following set of equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (9)$$

$$\frac{d\chi}{dt} = 0, \quad (10)$$

$$\frac{d\eta}{dt} = 0, \quad (11)$$

$$\frac{d\nu}{dt} = \frac{1}{2}\vec{v}^2 - w, \quad (12)$$

in which w is the specific enthalpy.

$$\frac{d\alpha}{dt} = \frac{\vec{\nabla}\eta \cdot \vec{J}}{\rho}, \quad (13)$$

$$\frac{d\beta}{dt} = -\frac{\vec{\nabla}\chi \cdot \vec{J}}{\rho}. \quad (14)$$

In all the above equations \vec{B} is given by equation (6) and \vec{v} is given by equation (7). The mass conservation equation (3) is readily obtained. Now one needs to show that also equation (1) and equation (4) are satisfied.

It can be easily shown that provided that \vec{B} is in the form given in equation (6), and equation (10) and equation (11) are satisfied, then equations (1) are satisfied.

We shall now show that a velocity field given by equation (7), such that the equations for $\alpha, \beta, \chi, \eta, \nu$ satisfy the corresponding equations (9,10,11,12, 13,14) must satisfy Euler's equations. Let us calculate the material derivative of \vec{v} :

$$\frac{d\vec{v}}{dt} = \frac{d\vec{\nabla}\nu}{dt} + \frac{d\alpha}{dt}\vec{\nabla}\chi + \alpha\frac{d\vec{\nabla}\chi}{dt} + \frac{d\beta}{dt}\vec{\nabla}\eta + \beta\frac{d\vec{\nabla}\eta}{dt}. \quad (15)$$

It can be easily shown that:

$$\begin{aligned} \frac{d\vec{\nabla}\nu}{dt} &= \vec{\nabla}\frac{d\nu}{dt} - \vec{\nabla}v_k\frac{\partial\nu}{\partial x_k} = \vec{\nabla}\left(\frac{1}{2}\vec{v}^2 - w\right) - \vec{\nabla}v_k\frac{\partial\nu}{\partial x_k}, \\ \frac{d\vec{\nabla}\eta}{dt} &= \vec{\nabla}\frac{d\eta}{dt} - \vec{\nabla}v_k\frac{\partial\eta}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial\eta}{\partial x_k}, \\ \frac{d\vec{\nabla}\chi}{dt} &= \vec{\nabla}\frac{d\chi}{dt} - \vec{\nabla}v_k\frac{\partial\chi}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial\chi}{\partial x_k}. \end{aligned} \quad (16)$$

In which x_k is a Cartesian coordinate and a summation convention is assumed. Equations (10,11,12) were used in the above derivation. Inserting the result from equations (13,14,16) into equation (15) yields:

$$\begin{aligned} \frac{d\vec{v}}{dt} &= -\vec{\nabla}v_k\left(\frac{\partial\nu}{\partial x_k} + \alpha\frac{\partial\chi}{\partial x_k} + \beta\frac{\partial\eta}{\partial x_k}\right) + \vec{\nabla}\left(\frac{1}{2}\vec{v}^2 - w\right) \\ &\quad + \frac{1}{\rho}((\vec{\nabla}\eta \cdot \vec{J})\vec{\nabla}\chi - (\vec{\nabla}\chi \cdot \vec{J})\vec{\nabla}\eta) \\ &= -\vec{\nabla}v_kv_k + \vec{\nabla}\left(\frac{1}{2}\vec{v}^2 - w\right) + \frac{1}{\rho}\vec{J} \times (\vec{\nabla}\chi \times \vec{\nabla}\eta) \\ &= -\frac{\vec{\nabla}p}{\rho} + \frac{1}{\rho}\vec{J} \times \vec{B}. \end{aligned} \quad (17)$$

In which we have used both equation (6) and equation (7) in the above derivation. This of course proves that the barotropic Euler equations can be derived from the action given in equation (8) and hence all the equations of barotropic magnetohydrodynamics can be derived from the above action without restricting the variations in any way except on the relevant boundaries and cuts. The reader should take into account that the topology of the magnetohydrodynamic flow is conserved, hence cuts must be introduced into the calculation as initial conditions.

5 A Simpler Action for Barotropic Magnetohydrodynamics

Can we obtain a further reduction of barotropic magnetohydrodynamics? Can we formulate magnetohydrodynamics with less than the six functions $\alpha, \beta, \chi, \eta, \nu, \rho$? The answer is yes, in fact four functions χ, η, ν, ρ will suffice. To see this we may write the two equations (10,11) as equations for α, β that is:

$$\begin{aligned}\frac{d\chi}{dt} &= \frac{\partial\chi}{\partial t} + \vec{v} \cdot \vec{\nabla}\chi = \frac{\partial\chi}{\partial t} + (\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta) \cdot \vec{\nabla}\chi = 0, \\ \frac{d\eta}{dt} &= \frac{\partial\eta}{\partial t} + \vec{v} \cdot \vec{\nabla}\eta = \frac{\partial\eta}{\partial t} + (\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta) \cdot \vec{\nabla}\eta = 0,\end{aligned}\quad (18)$$

in which we have used equation (7). Solving for α, β we obtain:

$$\begin{aligned}\alpha[\chi, \eta, \nu] &= \frac{(\vec{\nabla}\eta)^2(\frac{\partial\chi}{\partial t} + \vec{\nabla}\nu \cdot \vec{\nabla}\chi) - (\vec{\nabla}\eta \cdot \vec{\nabla}\chi)(\frac{\partial\eta}{\partial t} + \vec{\nabla}\nu \cdot \vec{\nabla}\eta)}{(\vec{\nabla}\eta \cdot \vec{\nabla}\chi)^2 - (\vec{\nabla}\eta)^2(\vec{\nabla}\chi)^2} \\ \beta[\chi, \eta, \nu] &= \frac{(\vec{\nabla}\chi)^2(\frac{\partial\eta}{\partial t} + \vec{\nabla}\nu \cdot \vec{\nabla}\eta) - (\vec{\nabla}\eta \cdot \vec{\nabla}\chi)(\frac{\partial\chi}{\partial t} + \vec{\nabla}\nu \cdot \vec{\nabla}\chi)}{(\vec{\nabla}\eta \cdot \vec{\nabla}\chi)^2 - (\vec{\nabla}\eta)^2(\vec{\nabla}\chi)^2}.\end{aligned}\quad (19)$$

Hence α and β are not free variables any more, but depend on χ, η, ν . Moreover, the velocity \vec{v} now depends on the same three variables χ, η, ν :

$$\vec{v} = \vec{\nabla}\nu + \alpha[\chi, \eta, \nu]\vec{\nabla}\chi + \beta[\chi, \eta, \nu]\vec{\nabla}\eta. \quad (20)$$

Since \vec{v} is given now by equation (20) it follows that the two equations (10,11) are satisfied identically and need not be derived from a variational principle. The above equation can be somewhat simplified resulting in:

$$\begin{aligned}\vec{v} &= \vec{\nabla}\nu + \frac{1}{\vec{B}^2}[\frac{\partial\eta}{\partial t}\vec{\nabla}\chi - \frac{\partial\chi}{\partial t}\vec{\nabla}\eta + \vec{\nabla}\nu \times \vec{B}] \times \vec{B} \\ &= \frac{1}{\vec{B}^2}[(\frac{\partial\eta}{\partial t}\vec{\nabla}\chi - \frac{\partial\chi}{\partial t}\vec{\nabla}\eta) \times \vec{B} + \vec{B}(\vec{\nabla}\nu \cdot \vec{B})]\end{aligned}\quad (21)$$

Hence the velocity \vec{v} is partitioned naturally into two components one which is parallel to the magnetic field and another one which is perpendicular to it:

$$\begin{aligned}\vec{v} &= \vec{v}_\perp + \vec{v}_\parallel \\ \vec{v}_\perp &= \frac{1}{\vec{B}^2}(\frac{\partial\eta}{\partial t}\vec{\nabla}\chi - \frac{\partial\chi}{\partial t}\vec{\nabla}\eta) \times \vec{B}, \quad \vec{v}_\parallel = \frac{\vec{B}}{\vec{B}^2}(\vec{\nabla}\nu \cdot \vec{B}).\end{aligned}\quad (22)$$

Inserting the velocity representation (21) into equation (19) will lead to the result:

$$\begin{aligned}\alpha &= \frac{\vec{\nabla}\eta \cdot (\vec{B} \times (\vec{v} - \vec{\nabla}\nu))}{\vec{B}^2} \\ \beta &= -\frac{\vec{\nabla}\chi \cdot (\vec{B} \times (\vec{v} - \vec{\nabla}\nu))}{\vec{B}^2}.\end{aligned}\quad (23)$$

Finally equations (19) should be substituted into equation (8) to obtain a Lagrangian density \mathcal{L} in terms of χ, η, ν, ρ .

$$\begin{aligned}\mathcal{L}[\chi, \eta, \nu, \rho] &\equiv -\rho\left[\frac{\partial\nu}{\partial t} + \alpha[\chi, \eta, \nu]\frac{\partial\chi}{\partial t} + \beta[\chi, \eta, \nu]\frac{\partial\eta}{\partial t} + \varepsilon(\rho)\right] \\ &+ \frac{1}{2}(\vec{\nabla}\nu + \alpha[\chi, \eta, \nu]\vec{\nabla}\chi + \beta[\chi, \eta, \nu]\vec{\nabla}\eta)^2 \\ &- \frac{1}{8\pi}(\vec{\nabla}\chi \times \vec{\nabla}\eta)^2.\end{aligned}\quad (24)$$

Using equations (23) this can be written as:

$$\mathcal{L}[\chi, \eta, \nu, \rho] = \rho\left[\frac{1}{2}\vec{v}^2 - \frac{d\nu}{dt} - \varepsilon(\rho)\right] - \frac{1}{8\pi}\vec{B}^2 \quad (25)$$

where \vec{v} is given by equation (21) and \vec{B} by equation (6). Or more explicitly as:

$$\begin{aligned}\mathcal{L}[\chi, \eta, \nu, \rho] &= \frac{1}{2}\frac{\rho}{(\vec{\nabla}\chi \times \vec{\nabla}\eta)^2}\left[\vec{\nabla}\eta\frac{\partial\chi}{\partial t} - \vec{\nabla}\chi\frac{\partial\eta}{\partial t} + (\vec{\nabla}\chi \times \vec{\nabla}\eta) \times \vec{\nabla}\nu\right]^2 \\ &- \rho\left[\frac{\partial\nu}{\partial t} + \frac{1}{2}(\vec{\nabla}\nu)^2 + \varepsilon(\rho)\right] - \frac{(\vec{\nabla}\chi \times \vec{\nabla}\eta)^2}{8\pi}.\end{aligned}\quad (26)$$

This Lagrangian density admits an infinite symmetry group of transformations of the form:

$$\hat{\eta} = \hat{\eta}(\chi, \eta), \quad \hat{\chi} = \hat{\chi}(\chi, \eta), \quad (27)$$

provided that the absolute value of the Jacobian of these transformation is unity:

$$\left|\frac{\partial(\hat{\eta}, \hat{\chi})}{\partial(\eta, \chi)}\right| = 1. \quad (28)$$

In particular the Lagrangian density admits an exchange symmetry:

$$\hat{\eta} = \chi, \quad \hat{\chi} = \eta. \quad (29)$$

As a consequence of the double infinite symmetry group we have two *local* conservation laws given by the two equations (10,11). Taking the variational derivatives of the action defined using equation (26) to zero for arbitrary variations leads to the following set of equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (30)$$

$$\frac{d\nu}{dt} = \frac{1}{2} \vec{v}^2 - w, \quad (31)$$

$$\frac{d\alpha[\chi, \eta, \nu]}{dt} = \frac{\vec{\nabla} \eta \cdot \vec{J}}{\rho}, \quad (32)$$

$$\frac{d\beta[\chi, \eta, \nu]}{dt} = -\frac{\vec{\nabla} \chi \cdot \vec{J}}{\rho}. \quad (33)$$

Those equations should be solved for χ, η, ν, ρ . Equations (32,33) contain a complicated linear combination of the second derivatives $\frac{\partial^2 \chi}{\partial t^2}$ and $\frac{\partial^2 \eta}{\partial t^2}$. This is inconvenient numerically therefore the following approach is recommended. Taking the partial temporal derivative of the two equations (10,11) we obtain:

$$\frac{\partial^2 \chi}{\partial t^2} + \frac{\partial \vec{v}}{\partial t} \cdot \vec{\nabla} \chi + \vec{v} \cdot \vec{\nabla} \frac{\partial \chi}{\partial t} = 0, \quad \frac{\partial^2 \eta}{\partial t^2} + \frac{\partial \vec{v}}{\partial t} \cdot \vec{\nabla} \eta + \vec{v} \cdot \vec{\nabla} \frac{\partial \eta}{\partial t} = 0. \quad (34)$$

Using the expression $\frac{\partial \vec{v}}{\partial t}$ from equation (17) we obtain an explicit expression for the second derivatives of the form:

$$\begin{aligned} \frac{\partial^2 \chi}{\partial t^2} &= ((\vec{v} \cdot \vec{\nabla}) \vec{v} + \vec{\nabla} w - \frac{1}{\rho} \vec{J} \times \vec{B}) \cdot \vec{\nabla} \chi - \vec{v} \cdot \vec{\nabla} \frac{\partial \chi}{\partial t} \\ \frac{\partial^2 \eta}{\partial t^2} &= ((\vec{v} \cdot \vec{\nabla}) \vec{v} + \vec{\nabla} w - \frac{1}{\rho} \vec{J} \times \vec{B}) \cdot \vec{\nabla} \eta - \vec{v} \cdot \vec{\nabla} \frac{\partial \eta}{\partial t}. \end{aligned} \quad (35)$$

Hence we arrived at a four function formalism for barotropic magnetohydrodynamics which can be derived from a Lagrangian. Notice, however, that this formalism contains two first order equations and two second order equations, while our previous six function formalism [8] contained six first order equations.

6 Conclusion

We have shown that barotropic magnetohydrodynamics can be represented in terms of four scalar functions χ, η, ν, ρ instead of the seven quantities

which are the magnetic field \vec{B} the velocity field \vec{v} and the density ρ . Anticipated applications include stability analysis and the description of numerical schemes using the described variational principles, exceed the scope of this paper.

It was shown by the author [18] that variational principles can be used directly for numerical analysis (simulation) without the need to refer to the field equations. This mathematical construction may lead to better algorithms for simulating magnetohydrodynamics in terms of the needed computer memory and CPU time. This approach was applied to potential flows in a series of papers [19, 20, 21]. Moreover, it was implemented in a user friendly software package FLUIDEX (which can be down loaded from the web site www.fluidex-cfd.com). A variational formalism of magnetohydrodynamics should serve the same use.

As for stability analysis I suspect that for achieving this we will need to add additional constants of motion constraints to the action as was done by [22], hopefully this will be discussed in a future paper.

References

- [1] Sturrock P. A., *Plasma Physics*, Cambridge University Press, Cambridge, 1994.
- [2] Vladimirov V. A. and H. K. Moffatt, "On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part 1. fundamental principles," *J. Fluid. Mech.*, Vol. 283, 125–139, 1995.
- [3] Kats A. V., "Variational principle in canonical variables, Weber transformation, and complete set of the local integrals of motion for dissipation-free magnetohydrodynamics," (Los Alamos Archives physics-0212023) *JETP Lett*, Vol. 77, 657, 2003.
- [4] Kats A. V. and V. M. Kontorovich, "Hamiltonian description of the motion of discontinuity surfaces," *Low Temp. Phys.*, Vol. 23, 89, 1997.
- [5] Kats A. V., "Variational principle and canonical variables in hydrodynamics with discontinuities," *Physica D*, Vol. 459, 152-153, 2001.
- [6] Sakurai T., "A New Approach to the Force-Free Field and Its Application to the Magnetic Field of Solar Active Regions," *Pub. Ast. Soc. Japan*, Vol. 31, 209, 1979.

- [7] Yang W. H. , P. A. Sturrock and S. Antiochos, "Force-free magnetic fields- The magneto-frictional method," *Ap. J.*, Vol. 309, 383, 1986.
- [8] Yahalom A. and D. Lynden-Bell, "Simplified Variational Principles for Barotropic Magnetohydrodynamics," (Los-Alamos Archives - physics/0603128) *Journal of Fluid Mechanics*, Vol. 607, 235–265, 2008.
- [9] Yahalom A., "Using Magnetic Fields to Optimize Material Flow - a Variational Approach to Magnetohydrodynamics," in *Proceedings of the Israeli - Russian Bi-National Workshop*, Jerusalem, Israel, June 2007.
- [10] Frenkel A., E. Levich and L. Stilman, "Hamiltonian description of ideal MHD revealing new invariants of motion" *Phys. Lett. A*, Vol. 88, 461, 1982.
- [11] Zakharov V. E. and E. A. Kuznetsov, "Hamiltonian Formalism for Non-linear Waves" *Usp. Fiz. Nauk*, Vol. 40, 1087, 1997.
- [12] Yahalom A. and D. Lynden-Bell, "Simplified Variational Principles for Barotropic Fluid Dynamics," (Los-Alamos Archives - physics/ 0603162), submitted to *Journal of Fluid Mechanics*.
- [13] Yahalom A., "CFD Methods Derived from Simplified Variational Principles," in *Proceedings of the AIAA Conference*, Reno, USA, January 2007.
- [14] Yahalom A., "A Finite Element Approach Derived from the Simplified Variational Principle," in *Proceedings of the 9th ASME Engineering Systems Design and Analysis Conference (ESDA 2008)*, Haifa, Israel, July 2008.
- [15] Yahalom A., "Simplified Variational Principles for Stationary Barotropic Fluid Dynamics," in *Proceedings of the Fifth International Conference on Mathematical Modeling of Metal Technologies (MMT 2008)*, Ariel, Israel, September 2008.
- [16] Dungey J. W., *Cosmic Electrodynamics*, Cambridge University Press, Cambridge, 1958.
- [17] Lamb H., *Hydrodynamics (p. 248)*, Dover Publications, 1945.
- [18] Yahalom A., "Method and System for Numerical Simulation of Fluid Flow," US patent 6,516,292, 2003.

- [19] Yahalom A., & G. A. Pinhasi, "Simulating Fluid Dynamics using a Variational Principle," in *Proceedings of the AIAA Conference*, Reno, USA, January 2003.
- [20] Yahalom A., G. A. Pinhasi and M. Kopylenko, "A Numerical Model Based on Variational Principle for Airfoil and Wing Aerodynamics," in *Proceedings of the AIAA Conference*, Reno, USA, January 2005.
- [21] Ophir D., A. Yahalom, G. A. Pinhasi and M. Kopylenko, "A Combined Variational & Multi-grid Approach for Fluid Simulation," in *Proceedings of the International Conference on Adaptive Modelling and Simulation (ADMOS 2005)*, Barcelona, Spain, September 2005, 295-304.
- [22] Yahalom A., J. Katz & K. Inagaki "Energy Principles for Self-Gravitating Barotropic Flows-Part Two-the Stability of Maclaurin Discs," *Mon. Not. R. Astron. Soc.*, Vol. 268, 506–516, 1994.